

Improved basis set for low frequency plasma waves

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[1] It is shown that the low frequency plasma wave equation can be obtained much more directly than by the previously used method of solving for the determinant of a matrix involving the three components of the electric field vector. The more direct method uses a two-dimensional current density vector space that is precisely equivalent to the previously used three-dimensional electric field vector space. Unlike the electric field, the current density is restricted by the quasi-neutrality condition to a two-dimensional vector space. Comparison with previously obtained dispersion relations is provided and a method is presented for obtaining exact analytic solutions for the three roots of the cubic dispersion relation. The commonly used kinetic Alfvén dispersion relation is shown to be valid only for near-perpendicular propagation in a low beta plasma. It is shown that at a cross-over point where the perpendicular wave phase velocity equals the ion acoustic velocity, the coupling between Alfvén and fast modes vanishes and the Alfvén mode reverts to its cold form even in situations where the Alfvén velocity is smaller than the electron thermal velocity. A method is prescribed by which measurement of wave electric current density completely eliminates the space-time ambiguity previously believed to be an unavoidable shortcoming of single-spacecraft frequency measurements.

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1. Introduction

[2] Several apparently different approaches have been used to describe low frequency waves in a warm magnetized plasma. While all assume an $\sim \exp(ik_x x + ik_z z - i\omega t)$ wave dependence and an equilibrium magnetic field $\mathbf{B} = B\hat{z}$, it is not obvious how these approaches relate to each other. The algebraic manipulations used have varying complexity and surprising cancelations sometimes occur after much tedious algebra. This variation in complexity suggests that the different approaches are effectively using different basis sets in some multidimensional vector space to characterize the same physics. A non-optimum basis set would require more algebra to arrive at the same final result and fortuitous-appearing cancelations would result. On the other hand, if an optimum basis set were chosen, algebraic complexity would be minimized, no surprising cancelations would occur, and the underlying physics would be more transparent. The results of various previously used approaches are first summarized and then a derivation in an optimized basis set is presented. This is used to identify limitations and occasional inconsistencies or errors in the previous approaches. These previous approaches are the generalized Ohm's

law method discussed by *Stringer* [1963], the two-fluid method discussed by *Hollweg* [1999], the 2×2 matrix method used by *Hasegawa and Uberoi* [1982, section 2.8.3], *Morales and Maggs* [1997], and *Lysak and Lotko* [1996], and the 3×3 matrix kinetic method used by *Hirose et al.* [2004]. The cold plasma dispersion given by *Stix* [1992] is used as a reference when considering certain limiting situations. The previous approaches are all based on deriving a homogeneous vector equation involving the vector electric field and then obtaining a dispersion relation from this vector equation. The dispersion relations obtained using these various approaches are listed below:

1.1. Equation (4) of Stringer

[3] *Stringer* [1963] places no constraints on the ratio of wave phase velocity to particle thermal velocity and notes that the general wave equation has three high frequency roots where ions play an insignificant role and three low frequency roots where both ions and electrons are important. *Stringer* [1963] argues that the high frequency roots are conveniently eliminated by neglecting terms of order ω/kc and so obtains the dispersion relation

$$\begin{aligned} & \left(\cos^2 \theta - Q \frac{\omega^2}{k^2 v_A^2} \right) \left[\left(\cos^2 \theta - \frac{\omega^2}{k^2 c_S^2} \right) - Q \frac{\omega^2}{k^2 v_A^2} \left(1 - \frac{\omega^2}{k^2 c_S^2} \right) \right] \\ & = \left(1 - \frac{\omega^2}{k^2 c_S^2} \right) \frac{\omega^2}{\omega_{ci}^2} \cos^2 \theta. \end{aligned} \quad (1)$$

Here

$$Q = 1 + k^2 c^2 / \omega_{pe}^2 \quad (2)$$

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and $\cos \theta = k_z/k$. This dispersion relation has been used by *Formisano and Kennel* [1969] and by *Rogers et al.* [2001]. The derivation of equation (1) involves taking the determinant of a fully populated 3×3 matrix involving all three electric field components and is algebraically quite complicated [see *Stringer*, 1963, Appendix I; *Swanson*, 1989, section 3.3.1].

1.2. Hollweg Equation (38)

[4] *Hollweg* [1999] did not take a determinant, but did what is mathematically equivalent, namely manipulated a set of homogenous equations involving different unknowns until one homogenous equation in one unknown was obtained. Various approximations were invoked, and in particular some, but not all, terms of order ω^2/ω_{ci}^2 were dropped. The resulting dispersion relation was Hollweg's equation (38), namely

$$\left(\frac{\omega^2}{k_z^2 v_A^2} - 1\right) [\omega^2(\omega^2 - k^2 v_A^2) - \beta k^2 v_A^2 (\omega^2 - k_z^2 v_A^2)] = \omega^2 (\omega^2 - k^2 v_A^2) k_x^2 \left(\frac{c_s^2}{\omega_{ci}^2} - \frac{c^2}{\omega_{pe}^2} \frac{\omega^2}{k_z^2 v_A^2}\right) \quad (3)$$

where $\beta = c_s^2/v_A^2$. Equation (3) was claimed to be valid in the regime $\omega \ll \omega_{ci}$ and for arbitrary c_s/v_A and for arbitrary k_z/k . Because the algebraic details leading to equation (3) were not given but rather only characterized as being based on “three lengthy relations involving the three components of $\delta \mathbf{E}$ ”, it is not possible to replicate with certainty the manner in which equation (3) was derived.

1.3. The 2×2 Matrix Approach

[5] *Hasegawa and Uberoi* [1982], *Morales and Maggs* [1997], and *Lysak and Lotko* [1996] argued that the 3×3 matrix equation describing warm plasma waves could be approximated by a 2×2 matrix because the compressional (i.e., fast) mode could be factored out. This resulted in

$$\begin{bmatrix} \frac{\omega^2}{k_z^2 v_A^2} - 1 & \frac{k_x}{k_z} \\ \frac{k_x}{k_z} & \left(\frac{\omega^2}{k_z^2 c_s^2} - 1\right) \frac{\omega_{ci}^2}{k_z^2 v_A^2} - \frac{k_x^2}{k_z^2} \end{bmatrix} \cdot \begin{bmatrix} \tilde{E}_x \\ \tilde{E}_z \end{bmatrix} = 0 \quad (4)$$

the determinant of which in the limit $\omega^2 \gg k_z^2 c_s^2$ reduces to the well-known kinetic Alfvén dispersion relation

$$\omega^2 = k_z^2 v_A^2 \left(1 + \frac{k_x^2 c_s^2}{\omega_{ci}^2}\right). \quad (5)$$

Equation (5) was claimed to be valid in the regime $\omega \ll \omega_{ci}$ and $v_{Te} \gg \omega/k_z \gg v_{Ti}$ with no restriction on propagation angle.

1.4. The 3×3 Matrix Method Used by Hirose With Cold Ion Assumption

[6] *Hirose et al.* [2004] evaluated the components of the dielectric tensor calculated using kinetic theory and obtained an extremely complicated expression (44 terms) when both

ions and electrons were warm. However, when the ions were assumed cold, the expression reduced to

$$\begin{bmatrix} \frac{\omega^2}{k_z^2 v_A^2} \frac{1}{1 - \frac{\omega^2}{\omega_{ci}^2}} - 1 & i \frac{\omega^2}{k_z^2 v_A^2} \frac{\omega/\omega_{ci}}{1 - \frac{\omega^2}{\omega_{ci}^2}} & \frac{k_x}{k_z} \\ -i \frac{\omega^2}{k_z^2 v_A^2} \frac{\omega/\omega_{ci}}{1 - \frac{\omega^2}{\omega_{ci}^2}} & \frac{\omega^2}{k_z^2 v_A^2} \frac{1}{1 - \frac{\omega^2}{\omega_{ci}^2}} - \frac{k^2}{k_z^2} & -\frac{i\omega_{ci}}{\omega} \frac{\omega^2}{k_z^2 v_A^2} \frac{k_x}{k_z} \\ \frac{k_x}{k_z} & \frac{i\omega_{ci}}{\omega} \frac{\omega^2}{k_z^2 v_A^2} \frac{k_x}{k_z} & \left(\frac{\omega^2}{k_z^2 c_s^2} - 1\right) \frac{\omega_{ci}^2}{k_z^2 v_A^2} - \frac{k_x^2}{k_z^2} \end{bmatrix} \cdot \begin{bmatrix} \tilde{E}_x \\ \tilde{E}_y \\ \tilde{E}_z \end{bmatrix} = 0. \quad (6)$$

Evaluating the determinant of the matrix in equation (6) resulted in the amazing result that a quantity $(1 - \omega^2/\omega_{ci}^2)$ factored out exactly, resulting in the much simpler than expected dispersion relation

$$\begin{aligned} & (\omega^2 - k_z^2 v_A^2) (\omega^4 - \omega^2 k^2 (c_s^2 + v_A^2) + k^2 v_A^2 k_z^2 c_s^2) \\ &= \frac{k^2 c^2}{\omega_{pi}^2} \omega^2 v_A^2 k_z^2 (\omega^2 - k^2 c_s^2). \end{aligned} \quad (7)$$

1.5. Chen and Wu Polynomial Method

[7] *Chen and Wu* [2011a] wrote the complete set of two-fluid equations, assumed $\omega \ll \omega_{ci}$, and obtained a matrix equation involving the components of the electric field. By taking the determinant of this matrix, they obtained a cubic equation in $\Omega = \omega/k_z v_A$. The coefficients of the cubic equation were polynomials involving various dimensionless ratios such as the electron to ion mass ratio, $k_\perp^2 c_s^2/\omega_{ci}^2$, and $k_\perp^2 c^2/\omega_{pe}^2$.

1.6. Stix Cold Plasma Method

[8] For reference, the *Stix* [1992] analysis of cold plasma waves provides

$$\begin{bmatrix} S - n_z^2 & -iD & n_x n_z \\ iD & S - n^2 & 0 \\ n_x n_z & 0 & P - n_x^2 \end{bmatrix} \cdot \begin{bmatrix} \tilde{E}_x \\ \tilde{E}_y \\ \tilde{E}_z \end{bmatrix} = 0 \quad (8)$$

where $\mathbf{n} = c\mathbf{k}/\omega$, $S = 1 - \sum \omega_{p\sigma}^2/(\omega^2 - \omega_{c\sigma}^2)$, $D = \sum (\omega_{c\sigma}/\omega) \omega_{p\sigma}^2/(\omega^2 - \omega_{c\sigma}^2)$, $P = 1 - \sum \omega_{p\sigma}^2/\omega^2$. Evaluation of the determinant of the matrix in equation (8) gives

$$(S \sin^2 \theta + P \cos^2 \theta) (ck/\omega)^4 - [RL \sin^2 \theta + PS(1 + \cos^2 \theta)] (ck/\omega)^2 + PRL = 0 \quad (9)$$

where $R = S + D$ and $L = S - D$. Equation (9) is valid for all ω and k provided $\omega/k_z \gg v_{Te}$, v_{Ti} and $k_x r_{Le,i} \ll 1$. Equation (9) predicts $k \rightarrow 0$ when any one of P , R , or L vanish (cutoffs) and predicts that $k \rightarrow \infty$ for perpendicular propagation when $S \rightarrow 0$ (wave resonance).

1.7. Summary of Main Results

[9] This paper will demonstrate the following:

[10] 1. By using $\tilde{\mathbf{J}}$ instead of $\tilde{\mathbf{E}}$ as the fundamental quantity, the Stringer result can be derived in a quicker and more intuitive way than in the original paper.

[11] 2. Equation (3) involves an inconsistent retention of terms of order ω^2/ω_{ci}^2 and, contrary to the assertions in *Hollweg* [1999], is valid only if both $c_s \ll v_A$ and propagation is nearly perpendicular to the magnetic field.

[12] 3. Equation (7) is in agreement with the small ω/ω_{ci} limit of equation (1).

[13] 4. Equation (4) and hence equation (5) are only correct for $\cos^2\theta \ll \beta$, i.e., only for near-perpendicular propagation in a low β plasma.

[14] 5. Exact analytic solutions of the form $\omega = \omega(\mathbf{k})$ for the three different roots of the Stringer dispersion relation are given. These solutions are valid for arbitrary β and arbitrary propagation angle.

[15] 6. When $\omega^2/k^2c_s^2 = 1$, the Alfvén mode decouples from the fast mode and reverts to its cold plasma character even if $m_e/m_i < \beta \ll 1$; this is contrary to the prediction of the 2×2 matrix method and may have some geophysical implications.

[16] 7. Besides describing Alfvén and fast modes, the Stringer result describes the magnetized ion acoustic dispersion relation, the cold ion cyclotron wave dispersion, the lower hybrid resonance in a plasma where $\omega_{pe}^2 \gg \omega_{ce}^2$, and whistler waves.

[17] 8. A method is prescribed by which measurement of $\tilde{\mathbf{J}}$ completely eliminates the space-time ambiguity previously believed to be an unavoidable shortcoming of single-spacecraft frequency measurements.

[18] 9. The effect of finite resistivity (electron-ion collisions) is discussed and it is shown that the modes with $\omega^2 \ll \omega_{ci}^2$ are relatively unaffected by resistivity.

[19] 10. The relationship to the high-frequency quasi-longitudinal and quasi-transverse modes predicted by the Altar-Appleton-Hartree dispersion relation is discussed.

2. Derivation of Stringer Dispersion Using Optimum Vector Space

[20] The derivation of equation (1) by *Stringer* [1963] involves lengthy algebraic manipulations that eventually produce three homogeneous equations in the three unknowns $\mathbf{k} \cdot \tilde{\mathbf{E}}$, $\tilde{\mathbf{E}} \cdot \mathbf{B}$, and $\tilde{\mathbf{E}} \cdot \mathbf{k} \times \mathbf{B}$. The vanishing of the determinant of the coefficients of these three equations provides the dispersion relation given in equation (1). We show here that equation (1) can be obtained much more directly by using a different and more natural vector space, namely $\mathbf{k} \cdot \tilde{\mathbf{J}}$, $\mathbf{k}_\perp \cdot \tilde{\mathbf{J}}$, and $\mathbf{k} \times \mathbf{B} \cdot \tilde{\mathbf{J}}$ where $\tilde{\mathbf{J}} = \sum n_\sigma q_\sigma \tilde{\mathbf{u}}_\sigma$ is the electric current density associated with the wave. This vector space has the immediate advantage that $\mathbf{k} \cdot \tilde{\mathbf{J}} = 0$ for a quasi-neutral plasma so the system reduces to just two coupled equations in the two unknowns $\mathbf{k}_\perp \cdot \tilde{\mathbf{J}}$, and $\mathbf{k}_\perp \times \mathbf{B} \cdot \tilde{\mathbf{J}}$. The quasi-neutral set of equations comprise the equations of motion and continuity for each species σ ,

$$-i\omega m_\sigma n \tilde{\mathbf{u}}_\sigma = n q_\sigma (\tilde{\mathbf{E}} + \tilde{\mathbf{u}}_\sigma \times \mathbf{B}) - \gamma_\sigma \kappa T_\sigma i \mathbf{k} \tilde{n}, \quad (10)$$

$$-i\omega \tilde{n} + i \mathbf{k} \cdot \tilde{\mathbf{u}}_\sigma n = 0 \quad (11)$$

where $\gamma_\sigma = 1$ if $\omega/k_z \ll v_{T\sigma}$ (isothermal equation of state) and $\gamma_\sigma = 3$ if $\omega/k_z \gg v_{T\sigma}$ (adiabatic equation of state) together with Faraday's law

$$i \mathbf{k} \times \tilde{\mathbf{E}} = i \omega \tilde{\mathbf{B}} \quad (12)$$

and the pre-Maxwell Ampere's law

$$i \mathbf{k} \times \tilde{\mathbf{B}} = \mu_0 \tilde{\mathbf{J}}. \quad (13)$$

The pre-Maxwell Ampere's law provides the quasi-neutrality condition $\mathbf{k} \cdot \tilde{\mathbf{J}} = 0$ which is the critical assumption that enables the method presented here. As shown in Appendix A, the assumption of quasi-neutrality fails at the cold plasma $L = 0$ cutoff; this failure places the maximum frequency at which the quasi-neutrality assumption is valid at a value well above the ion cyclotron frequency.

[21] Combination of Faraday's law and the pre-Maxwell Ampere's law gives

$$\mathbf{k} \times (\mathbf{k} \times \tilde{\mathbf{E}}) = -i\omega \mu_0 \tilde{\mathbf{J}}. \quad (14)$$

Defining the one-fluid quantities $\tilde{\mathbf{U}} = (\sum m_\sigma n \mathbf{u}_\sigma) / \sum m_\sigma n$ and $\rho = \sum n m_\sigma$, and then summing equations (10) over species gives the one-fluid equation of motion

$$-i\omega \rho \tilde{\mathbf{U}} = \tilde{\mathbf{J}} \times \mathbf{B} - i \mathbf{k} \sum \gamma_\sigma \kappa T_\sigma \tilde{n}. \quad (15)$$

It should be noted that this one-fluid model differs in a very subtle and normally insignificant way from magnetohydrodynamics in terms of how temperature is defined. Specifically, the electron temperature used in (15) is the two-fluid temperature and so is defined in terms of the random electron velocities relative to the mean electron velocity. Similarly, the ion temperature is defined in terms of the random ion velocities relative to the mean ion velocity. In contrast, in magnetohydrodynamics the electron and ion temperatures are each defined with respect to the center of mass velocity of the entire plasma [see *Bellan*, 2006, section 2.6.2]. If the electrons and ions have the same mean velocity, there is no difference between the two-fluid and magnetohydrodynamic definitions of temperature.

[22] Multiplication of equation (11) by m_σ and summing over species gives

$$-i\omega \tilde{n} \sum m_\sigma + i \mathbf{k} \cdot \tilde{\mathbf{U}} \rho = 0 \quad (16)$$

which can be solved to give

$$\tilde{n} = \frac{\mathbf{k} \cdot \tilde{\mathbf{U}} \rho}{\omega \sum m_\sigma} = \frac{n}{\omega} \mathbf{k} \cdot \tilde{\mathbf{U}}. \quad (17)$$

Dotting equation (15) with $i \mathbf{k} \omega / \rho$ gives

$$\omega^2 \mathbf{k} \cdot \tilde{\mathbf{U}} = i \omega \mathbf{k} \cdot \frac{\tilde{\mathbf{J}} \times \mathbf{B}}{\rho} + k^2 \mathbf{k} \cdot \tilde{\mathbf{U}} \sum \frac{\gamma_\sigma n \kappa T_\sigma}{\rho}. \quad (18)$$

Solving for $\mathbf{k} \cdot \tilde{\mathbf{U}}$ and using $\rho/n \simeq m_i$ gives

$$\mathbf{k} \cdot \tilde{\mathbf{U}} = i \omega \frac{\mathbf{k} \cdot \tilde{\mathbf{J}} \times \mathbf{B}}{\rho(\omega^2 - k^2 c_s^2)} \quad (19)$$

where $c_s^2 = \sum \gamma_\sigma \kappa T_\sigma / m_i$ defines the sound velocity. Equation (17) then becomes

$$\frac{\tilde{n}}{n} = \frac{i \mathbf{k} \cdot \tilde{\mathbf{J}} \times \mathbf{B}}{\rho(\omega^2 - k^2 c_s^2)}. \quad (20)$$

Using equation (20), equation (15) reduces to

$$\tilde{\mathbf{U}} = \frac{i}{\omega \rho} \left(\tilde{\mathbf{J}} \times \mathbf{B} + \mathbf{k} \frac{\mathbf{k} \cdot \tilde{\mathbf{J}} \times \mathbf{B} c_s^2}{\omega^2 - k^2 c_s^2} \right). \quad (21)$$

The generalized Ohm's law is obtained as follows: equation (10) is multiplied by $\mu_0 q_\sigma / m_\sigma$ and then summed over species, terms of order m_e / m_i are then discarded, and the approximation

$$\mathbf{U}_\perp = \frac{(m_i \tilde{\mathbf{u}}_i + m_e \tilde{\mathbf{u}}_e)_\perp}{m_i + m_e} \simeq \tilde{\mathbf{u}}_{i\perp} \quad (22)$$

is made so $\tilde{\mathbf{u}}_e \times \mathbf{B} \rightarrow \tilde{\mathbf{U}} \times \mathbf{B} + \tilde{\mathbf{J}} \times \mathbf{B} / n q_e$. The generalized Ohm's law, obtained from these operations and approximations, is

$$-i\omega \frac{c^2}{\omega_{pe}^2} \mu_0 \tilde{\mathbf{J}} = \tilde{\mathbf{E}} + \tilde{\mathbf{U}} \times \mathbf{B} + \frac{\tilde{\mathbf{J}}}{n q_e} \times \mathbf{B} - \gamma_e \mu_0 \frac{c^2}{\omega_{pe}^2} \frac{\kappa T_e}{m_e} q_e i \mathbf{k} \tilde{n}. \quad (23)$$

Substituting for $\tilde{\mathbf{U}}$ in equation (23) and solving for $\tilde{\mathbf{E}}$ gives

$$\begin{aligned} \tilde{\mathbf{E}} = & -i\omega \frac{c^2}{\omega_{pe}^2} \mu_0 \tilde{\mathbf{J}} - \frac{i}{\omega \rho} \left(\tilde{\mathbf{J}} \times \mathbf{B} + \mathbf{k} \frac{\mathbf{k} \cdot \tilde{\mathbf{J}} \times \mathbf{B} c_s^2}{\omega^2 - k^2 c_s^2} \right) \times \mathbf{B} - \frac{\tilde{\mathbf{J}}}{n q_e} \\ & \times \mathbf{B} + \gamma_e \mu_0 \frac{c^2}{\omega_{pe}^2} \frac{\kappa T_e}{m_e} q_e i \mathbf{k} \tilde{n}. \end{aligned} \quad (24)$$

Substituting for $\tilde{\mathbf{E}}$ in equation (14) using equation (24) gives the sought-after vector equation involving $\tilde{\mathbf{J}}$ only,

$$\begin{aligned} \mathbf{k} \times \left\{ \mathbf{k} \times \left[-i\omega \frac{c^2}{\omega_{pe}^2} \mu_0 \tilde{\mathbf{J}} - \frac{i}{\omega \rho} \left(\tilde{\mathbf{J}} \times \mathbf{B} + \mathbf{k} \frac{\mathbf{k} \cdot \tilde{\mathbf{J}} \times \mathbf{B} c_s^2}{\omega^2 - k^2 c_s^2} \right) \right] \right. \\ \left. \times \mathbf{B} - \frac{\tilde{\mathbf{J}}}{n q_e} \times \mathbf{B} \right\} = -i\omega \mu_0 \tilde{\mathbf{J}}. \end{aligned} \quad (25)$$

The pressure term in equation (24) was annihilated upon being crossed with \mathbf{k} so the only way electron pressure contributes is as a term in the one-fluid equation of motion. At this point, it is noted that $B^2 / \mu_0 \rho = v_A^2$, $B / \mu_0 n q_e = -\omega_{ci} c^2 / \omega_{pi}^2$, and the geometric mean frequency is $\omega_{gm}^2 = |\omega_{ce} \omega_{ci}| = \omega_{pe}^2 v_A^2 / c^2$ so equation (25) can be expressed as

$$\begin{aligned} \mathbf{k} \times \left\{ \mathbf{k} \times \left[\frac{\omega^2}{\omega_{gm}^2} \tilde{\mathbf{J}} + (\tilde{\mathbf{J}} \times \hat{\mathbf{z}}) \times \hat{\mathbf{z}} + \mathbf{k} \times \hat{\mathbf{z}} \frac{\mathbf{k} \cdot \tilde{\mathbf{J}} \times \hat{\mathbf{z}} c_s^2}{(\omega^2 - k^2 c_s^2)} \right. \right. \\ \left. \left. + \frac{i\omega}{\omega_{ci}} \tilde{\mathbf{J}} \times \hat{\mathbf{z}} \right] \right\} - \frac{\omega^2}{v_A^2} \tilde{\mathbf{J}} = 0. \end{aligned} \quad (26)$$

Expanding equation (26) gives

$$\begin{aligned} \mathbf{k} \left(-\mathbf{k}_\perp \cdot \tilde{\mathbf{J}}_\perp + \frac{i\omega}{\omega_{ci}} \mathbf{k}_\perp \cdot \tilde{\mathbf{J}}_\perp \times \hat{\mathbf{z}} \right) - k^2 \left(\frac{\omega^2}{\omega_{gm}^2} \tilde{\mathbf{J}} - \tilde{\mathbf{J}}_\perp + \mathbf{k}_\perp \right. \\ \left. \times \hat{\mathbf{z}} \frac{\mathbf{k}_\perp \cdot \tilde{\mathbf{J}}_\perp \times \hat{\mathbf{z}} c_s^2}{(\omega^2 - k^2 c_s^2)} + \frac{i\omega}{\omega_{ci}} \tilde{\mathbf{J}}_\perp \times \hat{\mathbf{z}} \right) - \frac{\omega^2}{v_A^2} \tilde{\mathbf{J}} = 0. \end{aligned} \quad (27)$$

Dotting equation (27) first with $\mathbf{k}_\perp \times \hat{\mathbf{z}}$ and then with \mathbf{k}_\perp gives two coupled equations involving $\mathbf{k}_\perp \cdot \tilde{\mathbf{J}}_\perp$ and $\mathbf{k}_\perp \cdot \tilde{\mathbf{J}}_\perp \times \hat{\mathbf{z}}$ which can be expressed in matrix form as

$$\begin{bmatrix} \frac{\omega^2}{k_\perp^2 v_A^2} \left(1 + \frac{k^2 v_A^2}{\omega_{gm}^2} \right) - 1 & \frac{i\omega}{\omega_{ci}} \\ -\frac{i\omega}{\omega_{ci}} & \frac{\omega^2}{k_\perp^2 v_A^2} \left(1 + \frac{k^2 v_A^2}{\omega_{gm}^2} \right) - 1 - \frac{k_\perp^2 c_s^2}{(\omega^2 - k^2 c_s^2)} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{k}_\perp \cdot \tilde{\mathbf{J}}_\perp \\ \mathbf{k}_\perp \cdot \tilde{\mathbf{J}}_\perp \times \hat{\mathbf{z}} \end{bmatrix} = 0. \quad (28)$$

The determinant of the above matrix gives the dispersion relation

$$\left[\frac{\omega^2}{k_\perp^2 v_A^2} \left(1 + \frac{k^2 v_A^2}{\omega_{gm}^2} \right) - 1 - \frac{k_\perp^2 c_s^2}{\omega^2 - k^2 c_s^2} \right] \left[\frac{\omega^2}{k_\perp^2 v_A^2} \left(1 + \frac{k^2 v_A^2}{\omega_{gm}^2} \right) - 1 \right] = \frac{\omega^2}{\omega_{ci}^2} \quad (29)$$

which is identical to equation (1) as can be seen by multiplying equation (29) through by $(\omega^2 / k^2 c_s^2 - 1) \cos^2 \theta$ and noting that $Q = 1 + k^2 c^2 / \omega_{pe}^2 = 1 + k^2 v_A^2 / \omega_{gm}^2$. In the limit where $\omega / \omega_{ci} \rightarrow 0$, it is seen that the Alfvén mode has $\mathbf{k}_\perp \cdot \tilde{\mathbf{J}}_\perp$ finite while the compressional mode has $\mathbf{k}_\perp \cdot \tilde{\mathbf{J}}_\perp \times \hat{\mathbf{z}}$ finite. The acoustic mode involves c_s^2 and so also involves $\mathbf{k}_\perp \cdot \tilde{\mathbf{J}}_\perp \times \hat{\mathbf{z}}$ being finite. The corresponding polarizations of the electric field are then found using equation (24) with equation (20). Essentially, we have solved for $\tilde{\mathbf{E}}$ as a function of $\tilde{\mathbf{J}}$ and used this in equation (14) whereas the traditional approach is to solve for $\tilde{\mathbf{J}}$ as a function of $\tilde{\mathbf{E}}$ and use this in equation (14); the method presented here immediately gives a 2×2 matrix whereas the traditional method gives a 3×3 matrix that after much algebra and many seemingly fortuitous cancellations reduces to the same dispersion relation as that presented here.

[23] For $\omega \ll \omega_{ci}$ equation (29) describes kinetic and inertial Alfvén waves, the fast mode, and magnetized ion acoustic waves having $k^2 \lambda_D^2 \ll 1$. For $\omega \sim \omega_{ci}$ equation (29) describes the ion cyclotron waves of cold plasma theory and the electrostatic ion cyclotron waves of warm plasma theory. For $\omega_{ce} \gg \omega > \omega_{ci}$ equation (29) describes the lower hybrid resonance for an over-dense plasma, and whistler waves. It fails to describe modes where quasi-neutrality is not satisfied, namely ion acoustic waves for which $k^2 \lambda_D^2 > 1$ and modes near the $L = 0$ cold plasma wave cutoff. The fact that the right hand side of equation (29) is ω^2 / ω_{ci}^2 shows that any model that purports to describe coupling between the Alfvén mode (2nd square bracket on left hand side) and the acoustic mode (embedded in first square bracket on left hand side)

while dropping terms of order ω^2/ω_{ci}^2 as in *Hollweg* [1999] where cannot be correct.

3. Comparison of Equation (29) to Hirose

[24] If $\omega^2/\omega_{gm}^2 \ll 1$ equation (29) reduces to

$$\left(\frac{\omega^2}{k_z^2 v_A^2} - 1\right) \left(\frac{\omega^2}{k^2 v_A^2} - 1 - \frac{k_\perp^2 c_s^2}{\omega^2 - k^2 c_s^2}\right) = \frac{\omega^2}{\omega_{ci}^2} \quad (30)$$

which is identical to equation (7) for the situation of cold ions and warm electrons (i.e., where $c_s^2 = \kappa T_e/m_i$); however, Equations (1) and (29) have the advantage of also being valid for the situation where both ions and electrons are warm. Thus, equations (1), (7), and (29) are mutually consistent. Equation (30) clearly shows that the right hand coupling term is of order ω^2/ω_{ci}^2 and so all terms of this order must be kept in any evaluation of the coupling.

4. Comparison of Hollweg Result to Hirose and Stringer

[25] Rearranging equation (3) and using $\beta v_A^2 = c_s^2$ gives

$$\begin{aligned} & \left(\frac{\omega^2}{k_z^2 v_A^2} - 1\right) \left(\frac{\omega^2}{k^2 v_A^2} - 1 - \frac{k_\perp^2 c_s^2}{(\omega^2 - k^2 c_s^2)}\right) \\ &= \frac{\omega^2}{\omega_{ci}^2} \frac{k_\perp^2 c_s^2}{k^2 v_A^2} \frac{(\omega^2 - k^2 v_A^2)}{(\omega^2 - k^2 c_s^2)} \left(1 - \frac{m_e}{m_i} \frac{\omega^2}{c_s^2 k_z^2}\right). \end{aligned} \quad (31)$$

which has the same left hand side as equation (30); i.e., the $Q = 1$ limit of equation (1). The respective right hand sides of equations (30) and (31) agree only if $k_x^2/k^2 \simeq 1$, $\omega^2 \ll k^2 v_A^2$, and $\omega^2 \ll k^2 c_s^2$. This requires the propagation angle to be nearly perpendicular to the magnetic field. Thus, *Hollweg's* assertion that equation (3) is valid at all angles and for all values of c_s/v_A is not correct. We believe that the reason equation (30) differs from equation (31) is that some terms of order ω^2/ω_{ci}^2 were retained in *Hollweg* [1999] while others were discarded.

5. Exact Roots of Equation (29)

[26] By defining

$$\xi = \frac{\omega^2}{k^2 v_A^2}, \quad \beta = \frac{c_s^2}{v_A^2}, \quad \Lambda = \frac{k_\perp^2 v_A^2}{\omega_{ci}^2}, \quad \alpha = \cos^2 \theta \quad (32)$$

equation (29) becomes

$$\left(\frac{\xi Q}{\alpha} - 1\right) \left(\xi Q - \frac{\xi - \beta \alpha}{\xi - \beta}\right) = \xi \Lambda. \quad (33)$$

This can be expressed as a cubic equation in ξ , namely

$$\xi^3 - A\xi^2 + B\xi - C = 0 \quad (34)$$

$$\begin{aligned} A &= \frac{Q + Q^2 \beta + Q\alpha + \alpha\Lambda}{Q^2} \\ B &= \alpha \frac{(1 + 2Q\beta + \Lambda\beta)}{Q^2} \\ C &= \frac{\alpha^2 \beta}{Q^2}. \end{aligned} \quad (35)$$

Equation (34) can be solved exactly for arbitrary θ by using a trigonometric substitution method given by *Nickalls* [1993] and previously used in the context of Alfvén waves by *Chen and Wu* [2011a, 2011b]; however the coefficients A , B , and C used by *Chen and Wu* [2011a, 2011b] differ from equation (35) here and so appear to be in error (note that in *Chen and Wu* [2011a, 2011b], the parameter Q is the electron to ion mass ratio which is negligible compared to unity and so can be dropped from the expressions in *Chen and Wu* [2011a, 2011b]).

[27] On defining

$$\begin{aligned} p &= \frac{3B - A^2}{3} \\ q &= \frac{9AB - 2A^3 - 27C}{27} \end{aligned} \quad (36)$$

the exact roots of equation (34) are

$$\xi_j = 2\sqrt{-\frac{p}{3}} \cos\left(\frac{1}{3} \cos^{-1}\left(\frac{3q}{2p} \sqrt{-\frac{3}{p}}\right) - \frac{2\pi}{3}j\right) + \frac{A}{3}, \quad j = 0, 1, 2. \quad (37)$$

These solutions to equation (34) are valid for arbitrary θ , ω , k , c_s , and v_A . Choosing $j = 0, 1$, or 2 gives the fast mode, Alfvén mode, and acoustic mode respectively. A polar plot of ξ versus angle produces a CMA-like (Clemmow-Mullaly Allis) plot for given values of β and Λ . These plots can be compared to simpler polynomial expansions such as equation (5). Figure 1 provides an example of a CMA-like plot of the three modes for $\Lambda = 0.4$ and $\beta = 0.4$; the solid lines are plots of $\omega^2/k^2 v_A^2$ versus θ for the three exact roots of the determinant as given by equation (37). The inner root (slowest) is the sound wave, the intermediate root is the Alfvén wave, and the outer root is the fast wave. The non-solid lines show for comparison various approximations discussed above and in the following text. In particular, the MHD Alfvén wave dispersion $\omega^2/k^2 v_A^2 = \cos^2 \theta$ is indicated by a line with short dashes (line immediately outside the solid line labeled ‘Alfvén’) while equation (5) is plotted as a dotted line slightly to the right of the MHD Alfvén wave dispersion (corresponding to the prediction of equation (5) that $\omega^2/k_z^2 v_A^2$ always exceeds unity). Figure 2 provides a zoomed-in view of the lower-left corner of Figure 1 and shows how equation (5) (dotted line) is always faster than the MHD mode (dashed line) and only agrees with the exact solution for $\theta \rightarrow \pi/2$. This detailed plot also shows that the exact Alfvén solution (solid line) is faster than the MHD solution as $\theta \rightarrow \pi/2$ but slower for θ less than $\pi/2$. The exact Alfvén plot also shows that for finite ω/ω_{ci} , propagation in

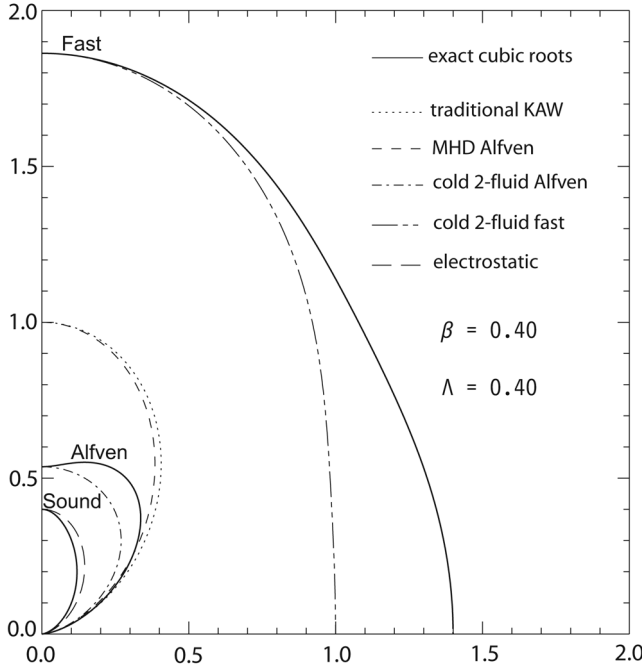


Figure 1. Plot of $\omega^2/k^2v_A^2$ versus angle. Vertical direction is $\theta = 0$ and horizontal direction is $\theta = \pi/2$; $\Lambda = 0.4$ and $\beta = 0.4$ have been used. Exact roots of cubic polynomial as given by equation (37) are heavy solid lines labeled ‘Sound’, ‘Alfven’, and ‘Fast’ respectively. For comparison, approximate solutions provided by various other models are shown from left to right as: long dashed line is electrostatic solution given by equation (57), dash-dot line is equation (63) with the minus sign chosen (cold 2-fluid slow, i.e., Alfvén mode), medium dashed line is the magnetohydrodynamic Alfvén dispersion $\omega^2/k^2v_A^2 = \cos^2\theta$, dotted line is traditional Kinetic Alfvén Wave dispersion given by equation (5), and dash-dot-dot line is cold 2-fluid fast wave given by equation (63) with plus sign chosen.

the parallel direction is slower than the Alfvén velocity and in fact, as the ion cyclotron frequency is approached from below, the parallel propagation velocity slows down to zero. This is in accordance with the $L = \infty$ cold plasma resonance whereby for parallel propagation $k \rightarrow \infty$ as $\omega \rightarrow \omega_{ci}$ from below.

[28] Equation (37) can be expressed as

$$\frac{\omega}{\omega_{ci}} = \sqrt{2\Lambda \sqrt{-\frac{p}{3}} \cos\left(\frac{1}{3} \cos^{-1}\left(\frac{3q}{2p} \sqrt{-\frac{3}{p}}\right) - \frac{2\pi j}{3}\right) + \frac{\Lambda A}{3}}, \quad j = 0, 1, 2 \quad (38)$$

which is an explicit dispersion relation of the form $\omega/\omega_{ci} = f(k_x c/\omega_{pi}, k_z c/\omega_{pi}, \beta)$ where β is a parameter, wave numbers are scaled to the ion skin depth c/ω_{pi} , and frequency is scaled to the ion cyclotron frequency.

6. Alfvén Solution

[29] In the low frequency limit $\omega/\omega_{ci} \rightarrow 0$ where the right hand side of equation (29) can be neglected, the left hand

side of equation (29) (and the equivalent, but more concise expression equation (33)) has two factors, with the root $\xi\alpha^{-1}Q - 1 = 0$ being the Alfvén mode. If ω/ω_{ci} is finite on the other hand, equation (33) can be re-arranged to be

$$\xi\alpha^{-1}Q - 1 = \frac{\xi\Lambda}{\xi Q - \frac{\xi - \beta\alpha}{\xi - \beta}}. \quad (39)$$

[30] We now consider waves with $\omega^2/k_z^2v_A^2$ being of order unity in a low β plasma. Since $\xi = \omega^2/k^2v_A^2 = \alpha$ this means that $\xi \gg \beta\alpha$ and so a factor ξ cancels from the right hand side of equation (39) which thus reduces to

$$\xi\alpha^{-1}Q - 1 = \frac{\Lambda}{Q - \frac{1}{\xi - \beta}}. \quad (40)$$

[31] We now restrict consideration to waves where the perpendicular wavelength is much shorter than the parallel wavelength, i.e., α is small, in which case $\xi \ll 1$ since ξ/α is of order unity. If $Q = 1 + k^2c^2/\omega_{pe}^2$ is of order unity, then because both ξ and β are small, $1/|\xi - \beta| \gg Q$ and so equation (40) further simplifies to

$$\xi\alpha^{-1}Q - 1 = \Lambda(\beta - \xi). \quad (41)$$

Solving for ξ gives

$$\frac{\xi}{\alpha} = \frac{1 + \Lambda\beta}{Q + \alpha\Lambda} = \frac{1 + \Lambda\beta}{1 + (m_e/m_i + \alpha)\Lambda}. \quad (42)$$

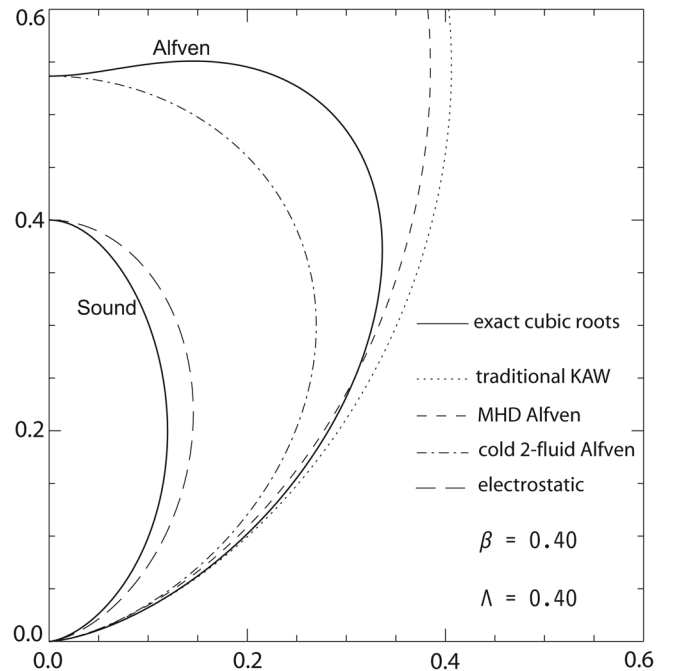


Figure 2. Close-up view of lower-left corner of Figure 1 showing how traditional KAW (dotted line) agrees with exact Alfvén solution (solid line) for near-perpendicular angle at which point both are slightly faster than MHD prediction (dashed line). However, at other angles, the classic KAW disagrees with exact solution which is much slower than MHD prediction.

[32] The traditional expression for the kinetic Alfvén wave, equation (5), corresponds to $\xi = \alpha(1 + \Lambda\beta)$ and examination of equation (41) shows this is valid only if $Q = 1$ (i.e., $k^2 c^2 / \omega_{pe}^2 \ll 1$) and $\xi \ll \beta$. Using equation (32) it is seen that equation (42) corresponds to

$$\frac{\omega^2}{k_z^2 v_A^2} = \frac{1 + k^2 c_s^2 / \omega_{ci}^2}{1 + (m_e/m_i + \cos^2\theta) \frac{k^2 v_A^2}{\omega_{ci}^2}} \quad (43)$$

or using $k^2 v_A^2 / \omega_{ci}^2 = k^2 c^2 / \omega_{pi}^2 = (m_i/m_e) k^2 c^2 / \omega_{pe}^2$

$$\frac{\omega^2}{k_z^2 v_A^2} = \frac{1 + (\beta m_i/m_e) k^2 c^2 / \omega_{pe}^2}{1 + \left(1 + (\beta m_i/m_e) \frac{\cos^2\theta}{\beta}\right) k^2 c^2 / \omega_{pe}^2}. \quad (44)$$

Equation (43) shows that the traditional kinetic Alfvén dispersion fails when $\cos^2\theta$ exceeds m_e/m_i and $k^2 v_A^2 / \omega_{ci}^2$ exceeds unity. Furthermore, if $\cos^2\theta$ exceeds β , then $\omega^2 / k_z^2 v_A^2$ becomes less than unity in contrast to the prediction of the traditional kinetic Alfvén wave dispersion that $\omega^2 / k_z^2 v_A^2$ exceeds unity. Equation (44) is to be contrasted to equation (3) of *Lysak and Lotko* [1996] which omits the term involving $(\cos^2\theta)/\beta$, i.e., is $\omega^2 / k_z^2 v_A^2 = \left(1 + (\beta m_i/m_e) k^2 c^2 / \omega_{pe}^2\right) / \left(1 + k^2 c^2 / \omega_{pe}^2\right)$. Incidentally, there appears to be an error in the Figure 1a contour plot in *Lysak and Lotko* [1996], since for $kc/\omega_{pe} = 10$ and $\beta m_i/m_e = 100$, it is seen that $\omega^2 / k_z^2 v_A^2 = \left(1 + (\beta m_i/m_e) k^2 c^2 / \omega_{pe}^2\right) / \left(1 + k^2 c^2 / \omega_{pe}^2\right) = (1 + 100 \times 100) / (1 + 100) = 99$ whereas the contour plot in question has $\omega^2 / k_z^2 v_A^2 \approx 9$. It appears that what was plotted in this figure was the erroneous quantity $\left(1 + (\beta m_i/m_e)^{1/2} k^2 c^2 / \omega_{pe}^2\right) / \left(1 + k^2 c^2 / \omega_{pe}^2\right)$.

[33] The decoupling of the Alfvén wave from the fast wave when $\xi = \beta$ corresponds to having $\omega^2 / k^2 = c_s^2$. This decoupling is evident in equation (18) where it is seen that $\mathbf{k} \cdot \tilde{\mathbf{J}} \times \mathbf{B}$ must vanish if $\omega^2 / k^2 = c_s^2$. This means that $\tilde{J}_y \rightarrow 0$ and since $\mu_0 \tilde{J}_y = ik_z \tilde{B}_x - ik_x \tilde{B}_z = -i(k_z^2 / k_x + k_x) \tilde{B}_z$ this means $\tilde{B}_z \rightarrow 0$.

7. Using $\tilde{\mathbf{J}}$ to Resolve the Space-Time Ambiguity of Single-Spacecraft Measurements

[34] By invoking $\tilde{\mathbf{J}}$ as the fundamental quantity, the analysis presented here differs from traditional analyses which invoked $\tilde{\mathbf{E}}$ as the fundamental quantity. Of course the same dispersion relation would be obtained no matter which of \tilde{n} , $\tilde{\mathbf{B}}$, or $\tilde{\mathbf{U}}$ is declared the fundamental quantity since all these quantities are proportional to each other. Nevertheless, choosing $\tilde{\mathbf{J}}$ results in a more transparent analysis because the determinant of a 2×2 matrix occurs rather than the determinant of a 3×3 matrix. The advantage of using $\tilde{\mathbf{J}}$ results from imposing quasi-neutrality at the beginning of the analysis rather than much later.

[35] This suggests that an advantage might also be incurred from using $\tilde{\mathbf{J}}$ in the analysis of measured quantities and, indeed, this turns out to be the case for the analysis of spacecraft observations where typically, a spacecraft moves at some velocity \mathbf{V}_{rel} relative to the plasma. The dispersion

relations derived here assume the observer is in the plasma rest frame but the frequency measured in the spacecraft frame is Doppler shifted by $\mathbf{k} \cdot \mathbf{V}_{rel}$ relative to the frequency observed in the plasma rest frame and also the spacecraft frame electric field differs from the plasma frame electric field by $\mathbf{V}_{rel} \times \mathbf{B}$. In particular, if a prime denotes a quantity measured in the spacecraft frame and no prime means a quantity measured in the plasma frame, one has the well-known relations

$$\omega' = \omega - \mathbf{k} \cdot \mathbf{V}_{rel} \quad (45)$$

and

$$\mathbf{E}' = \mathbf{E} + \mathbf{V}_{rel} \times \mathbf{B}. \quad (46)$$

It has traditionally been presumed that single spacecraft measurements cannot resolve how much of the observed ω' in equation (45) results from the ω term and how much from the $-\mathbf{k} \cdot \mathbf{V}_{rel}$ term. This space-time ambiguity of a single spacecraft has motivated the use of multispacecraft cluster missions.

[36] Spacecraft measurements have been mainly of $\tilde{\mathbf{E}}'$, $\tilde{\mathbf{B}}$, \tilde{n} and ω' with minimal attention to $\tilde{\mathbf{J}}$. We propose here a method whereby a single spacecraft measurement of $\tilde{\mathbf{J}}$ resolves the $\omega' \Leftrightarrow \omega$ space-time ambiguity. Because \mathbf{k} , $\tilde{\mathbf{B}}$ and $\tilde{\mathbf{J}}$ are frame-independent, the values of these quantities measured by a moving spacecraft are the same as what would be measured by an observer in the plasma frame. The equilibrium magnetic field unit vector \hat{z} is determined from $\hat{z} = \langle \mathbf{B}(t) \rangle / |\langle \mathbf{B}(t) \rangle|$ where angle brackets denote time-average. The wave number \mathbf{k} associated with any specific frequency must be orthogonal to both $\tilde{\mathbf{B}}$ and $\tilde{\mathbf{J}}$ associated with the same frequency since both $\mathbf{k} \cdot \tilde{\mathbf{B}} = 0$ and $\mathbf{k} \cdot \tilde{\mathbf{J}} = 0$; the former condition comes from the solenoidal property of magnetic fields and the latter condition comes from invocation of quasi-neutrality. This dual orthogonality condition establishes the unit wave vector to be

$$\hat{k} = \pm \frac{\tilde{\mathbf{B}} \times \tilde{\mathbf{J}}}{|\tilde{\mathbf{B}} \times \tilde{\mathbf{J}}|}. \quad (47)$$

[37] The wave vector magnitude k and resolution of the \pm ambiguity are obtained by invoking Faraday's law which is a frame-independent equation. Faraday's law in the spacecraft frame gives

$$k \hat{k} \times \tilde{\mathbf{E}}' = \omega' \tilde{\mathbf{B}}. \quad (48)$$

Substitution for \hat{k} in equation (48) using equation (47) gives

$$\pm k \frac{(\tilde{\mathbf{J}} \times \tilde{\mathbf{B}})}{|\tilde{\mathbf{J}} \times \tilde{\mathbf{B}}|} \times \tilde{\mathbf{E}}' = \omega' \tilde{\mathbf{B}}. \quad (49)$$

Either the plus or minus signs is selected in order to make the left hand side of equation (49) parallel to the right hand side. The magnitude k is then chosen so that the left hand side of equation (49) equals the right hand side. \mathbf{k} is thus fully determined from a single-spacecraft measurement and can then be used to calculate $\mathbf{k}_\perp \cdot \tilde{\mathbf{J}}_\perp$ and $\mathbf{k}_\perp \cdot \tilde{\mathbf{J}}_\perp \times \hat{z}$, the eigenvector component quantities in equation (28). The

relative velocity \mathbf{V}_{rel} is determined either from knowledge of the spacecraft orbit or from direct measurement of the mean vector velocity of the ions impacting the spacecraft. The plasma frame frequency ω is then determined unambiguously from equation (45) as $\omega = \omega' + \mathbf{k} \cdot \mathbf{V}_{rel}$; i.e., the space-time ambiguity has been overcome despite only a single spacecraft being used. We note that *Korepanov and Dudkin* [1999] proposed an alternate method to determine \mathbf{k} wherein Ampere's law rather than Faraday's law is invoked and both the magnitude and direction of $\tilde{\mathbf{J}}$ are used rather than just the direction.

[38] If it is found that the left and right hand sides of equation (49) are not in fact parallel, there would have to be one of the following: a measurement error, an error in the assumption of quasi-neutrality, or an error in the assumption that there is a only single \mathbf{k} associated with ω' . Thus, demonstrating parallelism of the right and left hand sides of equation (49) constitutes an effective validity check for the procedure.

[39] Substitution of equation (20) into equation (24) gives

$$\begin{aligned} \tilde{\mathbf{E}} = & -i\omega \frac{c^2}{\omega_{pe}^2} \mu_0 \tilde{\mathbf{J}} - \frac{i}{\omega \rho} \left(\tilde{\mathbf{J}} \times \mathbf{B} + \mathbf{k} \frac{\mathbf{k} \cdot \tilde{\mathbf{J}} \times \mathbf{B} c_s^2}{\omega^2 - k^2 c_s^2} \right) \times \mathbf{B} - \frac{\tilde{\mathbf{J}}}{nq_e} \\ & \times \mathbf{B} - \gamma_e \mu_0 \frac{c^2}{\omega_{pe}^2} \frac{\kappa T_e}{m_e} q_e \mathbf{k} \frac{\mathbf{k} \cdot \tilde{\mathbf{J}} \times \mathbf{B}}{\rho(\omega^2 - k^2 c_s^2)}. \end{aligned} \quad (50)$$

Equation (50) gives $\tilde{\mathbf{E}}$ as a function of $\tilde{\mathbf{J}}$, of $\{\omega, \mathbf{k}\}$, and of the equilibrium quantities ω_{pe}^2 , ρ , c_s^2 , \mathbf{B} , and T_e . This predicted $\tilde{\mathbf{E}}$ could be compared with the observed $\tilde{\mathbf{E}} = \tilde{\mathbf{E}}' - \mathbf{V}_{rel} \times \tilde{\mathbf{B}}$ as determined from equation (46). The quasineutrality assumption could be verified by demonstrating that $\varepsilon_0 |\mathbf{k} \cdot \tilde{\mathbf{E}}| / |q_e n| \ll 1$. By using a set of different frequencies ω' to determine associated values of \mathbf{k} and ω , a dispersion relation $\omega(\mathbf{k})$ would be established from measured quantities and then compared with the theoretical dispersion relation. Thus, measurement of $\tilde{\mathbf{J}}$ enables single spacecraft determination of all relevant wave properties with no space-time ambiguity.

[40] In the limit that electron inertia, Hall current, and warm plasma effects are negligible, equation (50) reduces to the MHD relation

$$\tilde{\mathbf{E}}_{\perp} \rightarrow -\frac{i}{\omega \rho} (\tilde{\mathbf{J}} \times \mathbf{B}) \times \mathbf{B} = \frac{iv_A^2}{\omega} \mu_0 \tilde{\mathbf{J}}_{\perp} \quad (51)$$

in which case the Alfvén mode has $\tilde{\mathbf{E}}_{\perp}$ parallel to \mathbf{k}_{\perp} while the fast mode has $\tilde{\mathbf{E}}_{\perp}$ parallel to $\mathbf{k}_{\perp} \times \hat{z}$. When any or all of electron inertia, Hall current, and warm plasma effects are significant, equation (50) shows that the Alfvén mode will have $\tilde{\mathbf{E}}_{\perp}$ deviate from being exactly parallel to \mathbf{k}_{\perp} and the fast mode will deviate from having $\tilde{\mathbf{E}}_{\perp}$ being exactly parallel to $\mathbf{k}_{\perp} \times \hat{z}$.

8. Some Geophysical Implications

[41] Examination of equation (41) shows that the cross-over from $\omega^2/k_z^2 v_A^2$ exceeding unity to being less than unity occurs when $\xi = \beta$ which, as shown above, corresponds to having $\omega^2/k^2 = c_s^2$. This cross-over is also evident upon examination of the right-hand side of equation (1) as this

right hand side reverses polarity when $\omega^2/k^2 c_s^2 = 1$ and so the polarity of any term due to coupling of the fast mode will reverse polarity. It should be noted that $\omega^2/k^2 = c_s^2$ is consistent with having $\omega^2/k_z^2 \gg c_s^2$ when $\cos^2 \theta$ is small.

[42] *Uritsky et al.* [2009] reported THEMIS measurements of auroral structures having perpendicular wavelengths $\lambda_x \sim 3 \times 10^5$ m and oscillation periods of $\tau \sim 10^2$ s, i.e., $\omega/k_x \sim 3 \times 10^3$ m s⁻¹. Since ionospheric plasma is primarily oxygen and the electron temperature is 1 eV, the ion acoustic velocity $c_s \sim 2.5 \times 10^3$ m s⁻¹ and so $\omega/k_x \sim c_s$. When mapped along a field the perpendicular wavelength increases as $(r/r_E)^{3/2}$, the temperature increases by about two orders of magnitude and the species changes from being primarily oxygen to being primarily hydrogen [*Lysak and Lotko*, 1996]. In particular, when mapping to a region where $r \sim 10r_E$, it is seen that $\omega^2/k^2 c_s^2 = f^2 \lambda_x^2 m_i / \kappa T_e$ will remain approximately constant since λ_x^2 increases by $\sim 10^3$, the ion mass drops by 16, and the electron temperature increases by about 10^2 . Hence, if the condition $\xi = \beta$ holds in the ionosphere, it will also be approximately true at large distances. In the ionosphere, $\beta m_i / m_e \ll 1$ so the Alfvén wave has a cold character (inertial Alfvén wave) whereas as shown by *Lysak and Lotko* [1996] at large distances from Earth, $\beta m_i / m_e \gg 1$ and it has been traditionally been presumed that if this is so the wave is described by equation (5). However, if $\xi = \beta$ remains approximately true over the length of the field line, then even though $\beta m_i / m_e \gg 1$, the wave dispersion will be approximately the same as the cold dispersion. As mentioned above, this can be seen from equation (1) where the right hand side vanishes when $\omega^2/k^2 c_s^2 = 1$ and so the Alfvén mode decouples from the fast mode; it can also be seen from equation (18). This has implications for the group velocity – as shown by *Morales and Maggs* [1997], the cold mode group velocity is much more confined to a field line than the group velocity associated with equation (5). This will give a tighter mapping of a disturbance at one point on a field line to an observer at another, distant point on the same field line.

9. Low Frequency, Electrostatic Limit

[43] The validity of equation (33) can be further checked by showing that it incorporates the quasi-neutral electrostatic limit, i.e., the electrostatic limit with $k^2 \lambda_{De}^2 \ll 1$. This limit corresponds to the magnetized acoustic mode and is retrieved by assuming $\xi Q \ll \alpha$ in which case $\xi Q \ll 1$ also; the assumption corresponds to $\omega^2 \ll k_z^2 v_A^2 / (1 + k^2 c^2 / \omega_{pe}^2)$. Equation (33) reduces to

$$\frac{\xi - \beta \alpha}{\xi - \beta} = \xi \Lambda \quad (52)$$

which can be recast as

$$\frac{\xi}{\beta} - \alpha - \frac{\alpha - 1}{\xi \Lambda - 1} = 0 \quad (53)$$

or, equivalently,

$$\frac{\omega^2}{k^2 c_s^2} - \cos^2 \theta - \frac{\omega^2 \sin^2 \theta}{\omega^2 - \omega_{ci}^2} = 0 \quad (54)$$

which is the magnetized plasma electrostatic ion acoustic wave dispersion relation with displacement current neglected. Equation (54) leads to a warm plasma resonance cone behavior with cone angle $\theta_c = \cos^{-1}(\omega/\omega_{ci})$ as observed experimentally in *Bellan* [1976]. Since the limit $\omega^2 \ll k_z^2 v_A^2$ corresponds to assuming $v_A \rightarrow \infty$ in Equations (3) and (30) and since the right hand term of equation (3) is quite different from the right hand side of equation (30) in this limit, it is clear that equation (3) (i.e., Hollweg's result) fails to reduce to the electrostatic limit. This distinction between the electrostatic limits of equations (3) and (30) further supports the claim that the right hand side of equation (33) is correct whereas the right hand side of equation (3) is not correct.

[44] If $\theta = \pi/2$, equation (54) becomes the electrostatic ion cyclotron wave

$$\omega^2 = \omega_{ci}^2 + k_x^2 c_s^2 \quad (55)$$

as discussed by *Stix* [1992, section 3–6, equation 59].

[45] If equation (52) is expressed as a quadratic in ξ

$$\xi^2 \Lambda - (1 + \beta \Lambda) \xi + \beta \alpha = 0 \quad (56)$$

then the ion acoustic mode is the root

$$\xi = \frac{(1 + \beta \Lambda) - \sqrt{(1 + \beta \Lambda)^2 - 4\beta \alpha \Lambda}}{2\Lambda} \quad (57)$$

which is plotted as the long dashed line in the lower left of Figure 1.

[46] The electrostatic ion cyclotron mode is the root

$$\xi = \frac{(1 + \beta \Lambda) + \sqrt{(1 + \beta \Lambda)^2 - 4\beta \alpha \Lambda}}{2\Lambda}. \quad (58)$$

10. Cold Plasma Ion Cyclotron Wave and Inertial Alfvén Wave

[47] In the limit $\xi \gg \beta$, the finite- β terms in equation (33) and hence the finite temperature terms in equation (29) may be dropped; this is the cold plasma limit.

[48] In the cold plasma limit equation (29) reduces to

$$\left[\left(\frac{\omega^2}{k^2 v_A^2} + \frac{\omega^2}{\omega_{gm}^2} \right) - 1 \right] \left[\frac{1}{\cos^2 \theta} \left(\frac{\omega^2}{k^2 v_A^2} + \frac{\omega^2}{\omega_{gm}^2} \right) - 1 \right] = \frac{\omega^2}{\omega_{ci}^2}. \quad (59)$$

If $\omega^2 \ll \omega_{gm}^2$ Equation (59) can be written as

$$n_x^2 = \frac{\left(\frac{c^2}{v_A^2} - n_z^2 \left(1 + \frac{\omega}{\omega_{ci}} \right) \right) \left(\frac{c^2}{v_A^2} - n_z^2 \left(1 - \frac{\omega}{\omega_{ci}} \right) \right)}{\frac{c^2}{v_A^2} - n_z^2 \left(1 - \frac{\omega^2}{\omega_{ci}^2} \right)} \quad (60)$$

where $n_x = ck_x/\omega$ and $n_z = ck_z/\omega$. Equation (60) is the cold plasma ion cyclotron wave dispersion given by *Stix* [1992, section 2–5, equation 19].

[49] If $\omega^2/\omega_{ci}^2 \ll 1$, then one of the roots of equation (59) is

$$\omega^2 = \frac{\cos^2 \theta}{\frac{1}{k^2 v_A^2} + \frac{1}{\omega_{gm}^2}} = \frac{k_z^2 v_A^2}{1 + \frac{k^2 c^2}{\omega_{pe}^2}} \quad (61)$$

which is the inertial Alfvén wave.

[50] Alternatively, equation (59) can be expressed as a quadratic equation. Ignoring finite β , equation (33) can be expressed as the quadratic

$$Q^2 \xi^2 - (Q + \alpha(\Lambda + Q)) \xi + \alpha = 0 \quad (62)$$

which has the solutions

$$\xi = \frac{Q + \alpha(\Lambda + Q) \pm \sqrt{(Q + \alpha(\Lambda + Q))^2 - 4\alpha Q^2}}{2Q^2}. \quad (63)$$

Choice of the minus sign gives the slow (i.e., Alfvén) mode while choice of the plus sign gives the fast mode. The slow mode is plotted as a dash-dot line in Figure 1 while the fast mode is plotted as a dash-dot-dot line.

11. Lower Hybrid Resonance of Over-Dense Plasma (Hybrid Resonance at ω_{gm})

[51] If $\xi \gg \beta$ then equation (33) becomes

$$\left(\frac{\xi Q}{\alpha} - 1 \right) (\xi Q - 1) = \xi \Lambda \quad (64)$$

If $\xi Q = \xi + 1$, which is equivalent to $\omega = \omega_{gm}$, equation (64) reduces to

$$\frac{\xi Q}{\alpha} = 1 + \Lambda \quad (65)$$

so if $\alpha \rightarrow 0$, ξ must also go to zero, corresponding to $k \rightarrow \infty$. Thus, the lower hybrid resonance (i.e., $S = 0$) in an over-dense plasma (i.e., $\omega_{pe}^2 \gg \omega_{ce}^2$) is retrieved since this resonance corresponds to $k \rightarrow \infty$ at $\omega = \omega_{gm}$ for perpendicular propagation.

12. High Frequency, Whistler Wave Limit

[52] If $\xi \gg 1$, $\xi \gg \beta$ and β is of order unity or smaller, then $(\xi - \beta\alpha)/(\xi - \beta) \rightarrow 1$. This corresponds to frequencies above the lower hybrid frequency since $\xi \gg 1$ whereas the lower hybrid frequency has $\xi \rightarrow 0$. The first terms in each of the parentheses in equation (33) dominate the other terms and so equation (33) reduces to

$$\xi = \frac{\Lambda \alpha}{Q^2}, \quad (66)$$

i.e.,

$$\omega^2 = \frac{k^4 v_A^4}{\omega_{ci}^2 \left(1 + k^2 c^2 / \omega_{pe}^2 \right)^2} \cos^2 \theta. \quad (67)$$

[53] This can be re-arranged as the whistler dispersion relation

$$n^2 = \frac{\omega_{pe}^2}{\omega(|\omega_{ce}|\cos\theta - \omega)} \quad (68)$$

where $n^2 = c^2 k^2 / \omega^2$ is the refractive index. If $k^2 c^2 / \omega_{pe}^2 \gg 1$, equation (67) reduces to

$$\omega^2 = \omega_{ce}^2 \cos^2 \theta \quad (69)$$

which gives the resonance cones of an over-dense (i.e., $\omega_{pe}^2 \gg \omega_{ce}^2$) cold plasma as described in *Fisher and Gould* [1969].

[54] The assumption that the first terms in each of the parentheses in equation (33) dominate the other terms corresponds to $\omega^2 / k_z^2 v_A^2$ and $\omega^2 / k^2 v_A^2$ dominating other terms in the matrix elements of equation (28) which thus reduces to

$$\begin{bmatrix} \frac{\omega^2}{k_z^2 v_A^2} \left(1 + \frac{k^2 c^2}{\omega_{pe}^2}\right) & \frac{i\omega}{\omega_{ci}} \\ -\frac{i\omega}{\omega_{ci}} & \frac{\omega^2}{k^2 v_A^2} \left(1 + \frac{k^2 c^2}{\omega_{pe}^2}\right) \end{bmatrix} \cdot \begin{bmatrix} \mathbf{k}_\perp \cdot \tilde{\mathbf{J}}_\perp \\ \mathbf{k}_\perp \cdot \tilde{\mathbf{J}}_\perp \times \hat{z} \end{bmatrix} = 0. \quad (70)$$

Since v_A^2 and ω_{ci} are both proportional to ion mass, the ion mass factors out of equation (70). In particular writing $v_A^2 / \omega_{ci} = -\omega_{ce} c^2 / \omega_{pe}^2$, equation (70) becomes

$$\begin{bmatrix} \frac{\omega}{k_z^2} \left(1 + \frac{k^2 c^2}{\omega_{pe}^2}\right) & -\frac{i\omega_{ce} c^2}{\omega_{pe}^2} \\ \frac{i\omega_{ce} c^2}{\omega_{pe}^2} & \frac{\omega}{k^2} \left(1 + \frac{k^2 c^2}{\omega_{pe}^2}\right) \end{bmatrix} \cdot \begin{bmatrix} \mathbf{k}_\perp \cdot \tilde{\mathbf{J}}_\perp \\ \mathbf{k}_\perp \cdot \tilde{\mathbf{J}}_\perp \times \hat{z} \end{bmatrix} = 0. \quad (71)$$

The dispersion relation shows that for small θ , $\left| \frac{\omega}{k_z^2} \left(1 + \frac{k^2 c^2}{\omega_{pe}^2}\right) \right|$ is of order of $\left| \frac{\omega_{ce} c^2}{\omega_{pe}^2} \right|$ in which case equation (71) implies that $\mathbf{k}_\perp \cdot \tilde{\mathbf{J}}_\perp$ is of order $\mathbf{k}_\perp \cdot \tilde{\mathbf{J}}_\perp \times \hat{z}$ but 90° out of phase, thereby demonstrating the helical nature of whistler propagation.

13. Demonstration That Equation (29) Is the Warm Plasma Modification of the Stix Cold Plasma Dispersion in the $\omega^2 \ll \omega_{ci}^2$ Limit

[55] In the limit $\omega^2 \ll \omega_{ci}^2$ it is seen that $D \rightarrow 0$ so cold plasma equation (8) simplifies to

$$\begin{bmatrix} S - n_z^2 & 0 & n_x n_z \\ 0 & S - n^2 & 0 \\ n_x n_z & 0 & P - n_x^2 \end{bmatrix} \cdot \begin{bmatrix} \tilde{E}_x \\ \tilde{E}_y \\ \tilde{E}_z \end{bmatrix} = 0. \quad (72)$$

which has the determinant

$$(S - n^2)(SP - n_x^2 S - n_z^2 P) = 0. \quad (73)$$

Additionally assuming $c^2 / v_A^2 \gg 1$ and $\omega^2 \ll \omega_{pe}^2$, it is seen that $S = c^2 / v_A^2$ and $P = -\omega_{pe}^2 / \omega^2$ so equation (73) can be expressed as

$$\left(\frac{c^2}{v_A^2} - \frac{c^2 k^2}{\omega^2} \right) \left(-\frac{c^2 \omega_{pe}^2}{v_A^2 \omega^2} - \frac{c^2 k_x^2}{\omega^2} \frac{c^2}{v_A^2} + \frac{c^2 k_z^2}{\omega^2} \frac{\omega_{pe}^2}{\omega^2} \right) = 0. \quad (74)$$

Upon multiplying by $(\omega^2 / k^2 c^2)(\omega^2 / k_z^2 c^2)(\omega^2 / \omega_{pe}^2)$ and re-arranging slightly this becomes

$$\left[\frac{\omega^2}{k^2 v_A^2} - 1 \right] \left[\frac{\omega^2}{k_z^2 v_A^2} \left(1 + \frac{k^2 c^2}{\omega_{pe}^2} \right) - 1 - \frac{\omega^2}{\omega_{gm}^2} \right] = 0. \quad (75)$$

Since it is assumed that $\omega^2 \ll \omega_{ci}^2$, it is clear that $\omega^2 \ll \omega_{gm}^2 = \omega_{ci}^2 m_i / m_e$ and so equation (75) reduces to

$$\left[\frac{\omega^2}{k^2 v_A^2} - 1 \right] \left[\frac{\omega^2}{k_z^2 v_A^2} \left(1 + \frac{k^2 c^2}{\omega_{pe}^2} \right) - 1 \right] = 0. \quad (76)$$

We also note that using $\omega_{gm}^2 = \omega_{pe}^2 v_A^2 / c^2$ and taking the limit $c_s^2 \rightarrow 0$, equation (29) reduces to

$$\left[\frac{\omega^2}{k^2 v_A^2} + \frac{\omega^2}{\omega_{gm}^2} - 1 \right] \left[\frac{\omega^2}{k_z^2 v_A^2} \left(1 + \frac{k^2 c^2}{\omega_{pe}^2} \right) - 1 \right] = \frac{\omega^2}{\omega_{ci}^2}. \quad (77)$$

If $\omega^2 \ll \omega_{ci}^2$ this becomes the same as equation (76) thereby demonstrating that equation (29) is the warm plasma extension of equation (9) and so involves the same fast and slow modes. The fast mode is the $\omega^2 / k^2 v_A^2 = 1$ root of equation (76) and the slow mode is the $\omega^2 / k_z^2 v_A^2 = 1 / (1 + k^2 c^2 / \omega_{pe}^2)$ root.

14. Effect of Finite Resistivity on Mode Properties

[56] The analysis so far has assumed zero-resistivity (perfect conductivity) but it is straightforward to generalize to include finite resistivity (finite conductivity). Resistivity results from the drag on electrons from their colliding with ions and is given by

$$\eta = \frac{m_e \nu_{ei}}{n q_e^2} \quad (78)$$

where ν_{ei} is the electron-ion collision frequency. Because the collision-induced drag of electrons on ions is equal and opposite to the collision-induced drag on ions by electrons, the summation of electron and ion momentum equations giving equation (15) results in a cancelation of these electron-on-ion and ion-on-electron drag forces. Thus finite resistivity does not change equation (15).

[57] In contrast, electron-ion collisions add an $\eta \tilde{\mathbf{J}}$ term to the right hand side of equation (23) which becomes

$$-i\omega \frac{c^2}{\omega_{pe}^2} \mu_0 \tilde{\mathbf{J}} = \tilde{\mathbf{E}} + \tilde{\mathbf{U}} \times \mathbf{B} - \eta \tilde{\mathbf{J}} + \frac{\tilde{\mathbf{J}}}{n q_e} \times \mathbf{B} - \gamma_e \mu_0 \frac{c^2}{\omega_{pe}^2} \frac{\kappa T_e}{m_e} q_e i \mathbf{k} \tilde{n}. \quad (79)$$

Since both the new resistive term and the existing electron inertia term on the left hand side are proportional to $\tilde{\mathbf{J}}$, the

two terms can be combined and placed on the left hand side giving

$$-i\omega \frac{c^2}{\omega_{pe}^2} \left(1 + \frac{i\nu_{ei}}{\omega}\right) \mu_0 \tilde{\mathbf{J}} = \tilde{\mathbf{E}} + \tilde{\mathbf{U}} \times \mathbf{B} + \frac{\tilde{\mathbf{J}}}{nq_e} \times \mathbf{B} - \gamma_e \mu_0 \frac{c^2}{\omega_{pe}^2} \frac{\kappa T_e}{m_e} q_e i \mathbf{k} \tilde{n}. \quad (80)$$

Thus, the effect of finite resistivity (or equivalently finite conductivity) is to replace $m_e \rightarrow m_e(1 + \frac{i\nu_{ei}}{\omega})$ since $\omega_{pe}^2 = nq_e^2/(\epsilon_0 m_e)$.

[58] The only place m_e appears in equation (29) is via the two terms involving $v_A^2/\omega_{gm}^2 = c^2/\omega_{pe}^2$. Thus finite resistivity (finite conductivity) is taken into account by simply letting $v_A^2/\omega_{gm}^2 \rightarrow (v_A^2/\omega_{gm}^2)(1 + i\nu_{ei}/\omega)$ in equation (29) which becomes

$$\left[\frac{\omega^2}{k^2 v_A^2} \left(1 + \frac{k^2 v_A^2}{\omega_{gm}^2} \left(1 + \frac{i\nu_{ei}}{\omega}\right)\right) - 1 - \frac{k_\perp^2 c_s^2}{\omega^2 - k^2 c_s^2} \right] \cdot \left[\frac{\omega^2}{k_z^2 v_A^2} \left(1 + \frac{k^2 v_A^2}{\omega_{gm}^2} \left(1 + \frac{i\nu_{ei}}{\omega}\right)\right) - 1 \right] = \frac{\omega^2}{\omega_{ci}^2}. \quad (81)$$

This shows that waves with $\omega \ll \omega_{ci}$ and dispersions $\omega^2 \sim k^2 v_A^2$ or $\omega^2 \sim k_z^2 v_A^2$ are quite insensitive to resistivity because for these waves $k^2 v_A^2/\omega_{gm}^2 \sim \omega^2/\omega_{gm}^2 \ll 1$ in which case the $i\nu_{ei}/\omega$ finite resistivity term multiplies what is already a small factor. The electron-ion collision frequency could thus be much larger than the wave frequency and yet still be unimportant. However, for waves where $k^2 v_A^2/\omega_{gm}^2 = k^2 c^2/\omega_{pe}^2$ is of order unity or larger, resistivity becomes important if the electron-ion collision frequency becomes of order the wave frequency. If $\nu_{ei} = \eta n q_e^2/m_e$ is large enough to make the imaginary part of ω as determined from equation (81) comparable in magnitude to the real part of ω , the wave will be heavily damped.

15. Connection of Low Frequency Modes to High Frequency Quasi-Transverse and Quasi-Longitudinal (Quasi-Parallel) Modes

[59] The Altar-Appleton-Hartree dispersion relation for high frequency modes also starts with the cold plasma dispersion given by equation (9), but then makes the assumption that all terms involving ions can be dropped from S , P , and D (see discussion on p. 37 of [Stix, 1992]). Assuming that ion terms can be dropped corresponds to assuming that the ions are infinitely massive and therefore stationary. Assuming infinite massive ions corresponds to assuming $\rho \rightarrow \infty$ in which case equation (21) gives $\tilde{\mathbf{U}} = 0$ and the cold plasma limit of the generalized Ohm's law, equation (23), becomes the electron equation of motion since $\tilde{\mathbf{J}}/nq_e$ is just the electron velocity. The electron equation of motion combined with Maxwell's equations and the assumptions that $\omega_{pe}^2/\omega^2 \ll 1$ and that displacement current can be neglected suffice to give the whistler dispersion in the form provided by equation (68) so it is not surprising that the one-fluid equation with generalized Ohm's law also describes whistler modes.

[60] Some additional aspects of the relationship between equation (29) the high-frequency dispersion will now be considered. In the high-frequency Altar-Appleton-Hartree model, certain simplifications result by either making Taylor expansions around $\theta = 0$ or around $\theta = \pi/2$. In particular, by making an expansion around θ nearly zero, quasi-longitudinal modes (also called QL or quasi-parallel modes) are determined while alternatively, by making an expansion around θ nearly $\pi/2$, quasi-transverse modes (also called QT modes) are determined. The QL modes are either right or left-handed circularly polarized (QLR, QLL) while the quasi-transverse are either ordinary (QTO, dispersion independent of magnetic field) or extraordinary (QTX, dispersion depends on magnetic field). The whistler wave is the high-frequency QLR mode in the $\omega^2 < \omega_{ce}^2, \omega_{pe}^2$ regime. As mentioned above, the fact that equation (29), a low frequency equation based on the $\omega^2 \ll \omega_{pe}^2, \omega_{ce}^2$ assumption also describes the whistler should not be surprising, as there is no reason why a wave cannot lie in the regime $\omega_{pi}^2, \omega_{gm}^2 \ll \omega^2 \ll \omega_{pe}^2, \omega_{ce}^2$ which indeed is the whistler regime. One might then ask whether equation (29) also describes high-frequency modes other than the whistler. Specifically, one can ask whether equation (29) describes the QLL, QTX, or QTO modes. The answer is yes in principle for the QLL modes and no for the QT modes. The reasons for these answers will now be given and furthermore, it will be shown that this ability of equation (29) to describe QLL modes and inability to describe QT modes is actually of negligible consequence because QLL, QTX, and QTO modes do not exist (i.e., do not propagate) in the $\omega_{pi}^2, \omega_{gm}^2 \ll \omega^2 \ll \omega_{pe}^2, \omega_{ce}^2$ quasi-neutral regime.

[61] The failure of equation (29) to describe high-frequency QT modes (if they were to exist) results from a violation of equation (22) for these modes. Since ions are at least three orders of magnitude heavier than electrons, it seems reasonable to assume that the center of mass velocity $\tilde{\mathbf{U}}$ is nearly the same as the ion velocity $\tilde{\mathbf{u}}_i$. This is certainly true at low frequencies where both electrons and ions are moving. However, for high-frequency waves the assumption that ion motion determines the center of mass velocity might be at variance with the assumption that ions are stationary and so the validity of equation (22) must be investigated in detail. Ions are unmagnetized in the regime $\omega_{ci} \ll \omega \ll \omega_{ce}$ and so have cold plasma motion

$$-i\omega m_i \tilde{\mathbf{u}}_i = q_i \tilde{\mathbf{E}} \quad (82)$$

while in this frequency regime electron perpendicular motion is given by

$$\tilde{\mathbf{u}}_e = \frac{\tilde{\mathbf{E}} \times \mathbf{B}}{B^2}. \quad (83)$$

Equation (22), the assumption that for the perpendicular direction the center of mass motion results from ion motion corresponds to assuming $m_i \tilde{\mathbf{u}}_{i\perp} + m_e \tilde{\mathbf{u}}_{e\perp} \simeq m_i \tilde{\mathbf{u}}_{i\perp}$ and this must be satisfied for each perpendicular component, i.e.,

$$m_i |\tilde{u}_{ix}| \gg m_e |\tilde{u}_{ex}| \quad (84a)$$

$$m_i |\tilde{u}_{iy}| \gg m_e |\tilde{u}_{ey}|. \quad (84b)$$

[62] Using equation (82) to give \tilde{u}_{ix} , \tilde{u}_{iy} and equation (83) to give \tilde{u}_{ex} , \tilde{u}_{ey} in equations (84a) and (84b) it is seen that validity of equation (22) requires

$$\left| \frac{q_i}{\omega} \tilde{E}_x \right| \gg m_e \left| \frac{\tilde{E}_y}{B} \right| \quad (85a)$$

$$\left| \frac{q_i}{\omega} \tilde{E}_y \right| \gg m_e \left| \frac{\tilde{E}_x}{B} \right|. \quad (85b)$$

If one assumes that $\cos^2\theta$ is nearly zero (i.e., a QT mode), equation (8) becomes approximately

$$\begin{bmatrix} S & -iD & 0 \\ iD & S - n^2 & 0 \\ 0 & 0 & P - n_x^2 \end{bmatrix} \cdot \begin{bmatrix} \tilde{E}_x \\ \tilde{E}_y \\ \tilde{E}_z \end{bmatrix} = 0 \quad (86)$$

and so the QTX mode (mode with dispersion $n^2 = (S^2 - D^2)/S$) has

$$\tilde{E}_x = i \frac{D}{S} \tilde{E}_y. \quad (87)$$

The high frequency QTO mode has dispersion $n_x^2 = P$ but this mode does not exist (propagate) for $\omega^2 < \omega_{pe}^2$ since P is negative in this frequency regime. Thus, all that matters is the QTX mode which is the mode involving the top two lines of equation (86) and having the polarization given by equation (87). Substituting equation (87) in equation (85b) gives the requirement

$$\left| \frac{\omega_{ce}}{\omega} \right| \gg \left| \frac{D}{S} \right|. \quad (88)$$

[63] We now determine D/S for the regime $\omega_{gm}^2, \omega_{pi}^2 \ll \omega^2 \ll \omega_{ce}^2$. The assumption of quasi-neutrality corresponds to dropping displacement current so

$$\begin{aligned} S &= 1 - \frac{\omega_{pi}^2}{\omega^2 - \omega_{ci}^2} - \frac{\omega_{pe}^2}{\omega^2 - \omega_{ce}^2} \\ &\rightarrow - \frac{\omega_{pi}^2}{\omega^2 - \omega_{ci}^2} - \frac{\omega_{pe}^2}{\omega^2 - \omega_{ce}^2} \end{aligned} \quad (89)$$

since the '1' in S corresponds to displacement current [see *Bellan*, 2006, p. 209, equations 6.9–6.12]. Then, assuming $\omega^2 \gg \omega_{gm}^2, \omega_{pi}^2$ and $\omega^2 \ll \omega_{ce}^2$ gives

$$S \simeq - \frac{\omega_{pe}^2}{\omega_{ce}^2}. \quad (90)$$

We note that assuming $\omega^2 \gg \omega_{gm}^2$ is a stronger condition than assuming $\omega^2 \gg \omega_{ci}^2$ and that the lower hybrid resonance at $\omega^2 = \omega_{gm}^2 = \omega_{ci}^2 m_i / m_e$ involves ion motion despite being at a frequency much higher than the ion cyclotron frequency. Evaluation of D in the regime $\omega^2 \gg \omega_{gm}^2, \omega_{pi}^2$ gives

$$D = \frac{\omega_{ci}}{\omega} \frac{\omega_{pi}^2}{\omega^2 - \omega_{ci}^2} + \frac{\omega_{ce}}{\omega} \frac{\omega_{pe}^2}{\omega^2 - \omega_{ce}^2} \rightarrow - \frac{\omega_{pe}^2}{\omega \omega_{ce}} = - \frac{\omega_{ce}}{\omega} S. \quad (91)$$

Substitution of equation (91) into equation (88) shows that equation (88) is clearly violated for the QTX mode. Thus, were they to exist QT modes could not be described by equation (29) in the regime $\omega_{gm}^2, \omega_{pi}^2 \ll \omega^2 \ll \omega_{ce}^2, \omega_{pe}^2$. This issue is of no practical consequence because the high-frequency wave equation shows that QT modes do not exist in this frequency regime except when displacement current is important but by virtue of the quasineutrality assumption we have restricted consideration to modes where displacement current can be neglected. The propagating QT modes that do depend on finite displacement current are in region 7 of the CMA diagram (CMA region numbers are defined in *Stix* [1992, Figure 2-1], *Swanson* [1989, Figure 2.8], and *Bellan* [2006, Figure 6.2]). This is a region where $L = 1 - \omega_{pe}^2 / (\omega(\omega + |\omega_{ce}|))$ is positive which clearly can only be true if displacement current (i.e., the '1' in L) is retained.

[64] Now let us turn attention to quasi-longitudinal waves in the $\omega_{gm}^2, \omega_{pi}^2 \ll \omega^2 \ll \omega_{ce}^2, \omega_{pe}^2$ regime. For quasi-longitudinal waves, terms involving $\sin\theta$ may be dropped while $\cos\theta$ is near-unity so equation (8) is approximately

$$\begin{bmatrix} S - n^2 & -iD & 0 \\ iD & S - n^2 & 0 \\ 0 & 0 & P \end{bmatrix} \cdot \begin{bmatrix} \tilde{E}_x \\ \tilde{E}_y \\ \tilde{E}_z \end{bmatrix} = 0 \quad (92)$$

and the wave dispersion is

$$(S - n^2)(S - n^2) - D^2 = 0 \quad (93)$$

which has the roots $n^2 = S \pm D = R, L$. Substituting these roots back into equation (92) shows that for quasi-longitudinal waves.

$$|\tilde{E}_x| = |\tilde{E}_y|. \quad (94)$$

In this case, each of equations (85a) and (85b) is individually satisfied because $\omega \ll \omega_{ce}$. Thus, the assumption $\tilde{u}_i = \tilde{\mathbf{U}}$ is valid for quasi-longitudinal waves in the $\omega_{gm}^2, \omega_{pi}^2 \ll \omega^2 \ll \omega_{ce}^2, \omega_{pe}^2$ regime. It thus is consistent that equation (29), an equation based on the $\tilde{\mathbf{u}}_i = \tilde{\mathbf{U}}$ assumption, describes whistler modes. In order to drop the '1' in S and have quasi-neutrality in this frequency regime, it is necessary to have $\omega_{pe}^2 \gg \omega_{ce}^2$, i.e., an over-dense plasma. There is no propagating QLL mode in regions 8 or 11 of the CMA diagram because L is negative in these regions as can be seen in *Bellan* [2006, Figure 6.2].

[65] We conclude that equation (29) reasonably describes propagating waves in regions 8, 11, and 13 of the CMA diagram. Equation (29) does not apply to the high frequency modes to the left of the $P = 0$ line in the CMA diagram because these all have $P > 0$ which violates the assumption $\omega^2 \ll \omega_{pe}^2$ used to derive equation (29). Similarly (29) does not apply to the modes below the $R = \infty$ (electron cyclotron resonance) line in the CMA diagram as these have $\omega^2 > \omega_{ce}^2$.

16. Comment Regarding Matrix Elements in *Lysak and Lotko* [1996]

[66] *Lysak and Lotko* [1996] argued that their matrix elements M_{yz} and M_{zy} could be neglected and so concluded that

their 3×3 matrix system involving E_x, E_y, E_z could be reduced to a 2×2 matrix system involving only E_x, E_z . In particular, they assert in their equation (A7) that their ε_{yz} matrix element contains the factor $\xi_{ns}Z(\xi_{ns})$ and cite *Chen* [1984] as the source for this matrix element. However, referral to *Chen* [1984, equation 7–143] shows that the ε_{yz} matrix element contains the factor $Z'(\xi_{ns})$ and not $\xi_{ns}Z(\xi_{ns})$. Since $Z'(\xi) = -2(1 + \xi Z(\xi))$ this is an important error since for small ξ , $Z(\xi) \simeq -2$ whereas for small ξ , $\xi Z(\xi) \simeq 0$. When this error is corrected, the electron term in *Lysak and Lotko* [1996, equation (A19)] gives

$$\varepsilon_{yz} = i \frac{\omega_{pe}^2}{\omega \omega_{ce}} \frac{k_{\perp}}{k_{\parallel}} = -i \frac{\omega_{pi}^2}{\omega \omega_{ci}} \frac{k_{\perp}}{k_{\parallel}} = -i \frac{\omega_{ci}}{\omega} \frac{c^2}{v_A^2} \frac{k_{\perp}}{k_{\parallel}}; \quad (95)$$

the ion term makes negligible contribution since $\xi_i \gg 1$. Because it is assumed when considering kinetic Alfvén waves that $\omega_{ci}/\omega \gg 1$, $c^2/v_A^2 \gg 1$, and $k_{\perp}/k_{\parallel} \gg 1$ the ε_{yz} term is significant and cannot be neglected. Thus, the argument in *Lysak and Lotko* [1996] for dropping the second line of their equation (A20) is invalid in which case their reduction of the 3×3 system to a 2×2 system is invalid. We note that equation (95) corresponds to equation (7) of *Hirose et al.* [2004] and to the yz term of the matrix in equation (6).

17. Conclusion

[67] The well-known form of the kinetic Alfvén wave, equation (5), is only valid when $\cos^2\theta < \beta$, a condition corresponding to near-perpendicular propagation for low β plasmas. For angles where $\cos^2\theta > \beta$, equation (5) is not correct, and the wave phase velocity is slower, not faster, than the phase velocity predicted by MHD. It is shown that if $\omega^2/k^2 c_s^2 = 1$, the Alfvén mode decouples from the fast mode and regains its cold plasma character even if $\beta m_i/m_e > 1$. The dispersion relation provided by *Hollweg* [1999] is only correct for near perpendicular propagation in low beta plasmas and because of inconsistent treatment of finite ω/ω_{ci} terms misses the critical behavior that occurs when $\omega^2/k^2 c_s^2 = 1$. By using components of the current density as the variables rather than the components of the electric field, the dispersion relation provided by *Stringer* [1963] is derived in a quicker and more transparent manner. Distinguishing modes by their polarization (in effect, by their eigenvectors) is more straightforward using the polarization of the current than polarization of electric field; this may be of practical use since it is often easier to measure the direction of currents than the direction of electric fields. The dispersion relation provided by *Hirose et al.* [2004] is in agreement with the *Stringer* dispersion relation in the low frequency regimes where both are expected to be valid. An analytic expression involving trigonometric functions gives the three exact roots of the cubic dispersion relation, i.e., gives $\omega = \omega(\mathbf{k})$ without approximation for each of the sound, Alfvén, and fast modes. These dispersion relations are exact providing quasi-neutrality is satisfied and cover the frequency range from well below the ion cyclotron frequency to the ion cyclotron frequency and above. A data analysis method involving single spacecraft measurement of $\tilde{\mathbf{J}}$ has been provided; this method completely eliminates the space-

time ambiguity previously believed to be an unavoidable shortcoming of single-spacecraft measurements.

Appendix A: Validity Range of Quasi-Neutrality Assumption

[68] For $\omega_{ce}^2 \gg \omega^2 \gg \omega_{pi}^2$, displacement current can be ignored only for plasmas having $\omega_{pe}^2/\omega_{ce}^2 \gg 1$ since in this regime $S = 1 + \omega_{pe}^2/\omega_{ce}^2$. For $\omega^2 \sim \omega_{ci}^2$ and for $\omega^2 \ll \omega_{ci}^2$, $S = 1 + \omega_{pi}^2/(\omega^2 - \omega_{ci}^2) + \omega_{pe}^2/\omega_{ce}^2$ and so the condition for ignoring displacement current is $\omega_{pi}^2/\omega_{ci}^2 \gg 1$ which corresponds to $c^2/v_A^2 \gg 1$ a much less stringent condition satisfied by most plasmas of interest. Ignoring displacement current is incorrect if there is a wave cutoff. This is because ignoring displacement current corresponds to assuming that $\omega/k \ll c$ which is clearly wrong if $k \rightarrow 0$. If $k \rightarrow 0$ then ω/k is clearly faster than any thermal velocity in which case cold plasma wave theory applies. The lowest frequency cutoff is the $L = 0$ cutoff which occurs when $\omega \simeq \omega_{ci} + \omega_{pe}^2/|\omega_{ce}|$; thus the dispersion relation derived here is valid so long as $\omega \ll \omega_{ci} + \omega_{pe}^2/|\omega_{ce}|$. If displacement current is important, then quasi-neutrality does not hold as can be seen by taking the divergence of the full Ampere's law. Since $\omega_{pe}^2/|\omega_{ce}| = \omega_{pi}^2/\omega_{ci}$ the validity condition can be expressed as

$$\omega \ll \omega_{ci} \left(1 + \frac{\omega_{pi}^2}{\omega_{ci}^2} \right) = \omega_{ci} \left(1 + \frac{c^2}{v_A^2} \right).$$

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